

Notes on a particular Weyl Algebra

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Abstract

By means of the notions of cross product algebras of the theory of quantum groups, in the context of classical Hopf algebra structures, we deduce some known structures of Weyl algebras type (as the Drinfeld quantum double, the restricted Heisenberg double, the generalized Schrödinger representation, and so on) that may be considered as a non-trivial examples of quantum groups having physical meaning, starting from a particular example of groupoid motivated by elementary quantum mechanics.

1. Introduction

In the paper [Iu], following a suggestion of Alain Connes (see [Co], I.1), it has been introduced a particular, simple groupoid, the so-called *Heisenberg-Born-Jordan EBB-groupoid* (or *HBJ EBB-groupoid*), whose physical motivations were, mainly, of spectroscopical nature.

An *E-groupoid* (in the notations of [Iu]) is an algebraic system of the type $(G, G^{(0)}, r, s, \star)$, with $G, G^{(0)}$ non-void sets, $G^{(0)} \subseteq G$, $G^{(0)}$ set of unities, $r, s : G \rightarrow G^{(0)}$ and $\star : G^{(2)} \rightarrow G$ partial groupoid law defined on $G^{(2)} = \{(g_1, g_2) \in G \times G, s(g_1) = r(g_2)\}$, satisfying the set of axioms described in [Iu], § 1.

The HBJ EBB-groupoid is a particular E-groupoid that has been denoted with $\mathcal{G}_{HBJ}(\mathcal{F}_I) = (\Delta\mathcal{F}_I, \mathcal{F}_I, r, s, \tilde{+})$, where $\mathcal{F}_I = \{\nu_i; \nu_i \in \mathbb{R}^+, i \in I \subseteq \mathbb{N}\}$ is the set of energy levels of a certain spectroscopic physical system, $\Delta\mathcal{F}_I = \{\nu_{ij}; \nu_{ij} = \nu_i - \nu_j, i, j \in I \subseteq \mathbb{N}\}$, $r : (i, j) \rightarrow i$ and $s : (i, j) \rightarrow j$ are the range and source maps, respectively, and $\nu_{ij} \tilde{+} \nu_{jk} = \nu_{ik}$ is the (partial) groupoid law as algebraic result of the Ritz-Rydberg combination principle.

In [Iu], it has been only considered the structure of a no finitely generated

groupoid algebra on $\mathcal{G}_{HBJ}(\mathcal{F}_I)$, say $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) = (\langle \Delta(\mathcal{F}_I) \rangle, +, \cdot, *)$, respect to an arbitrary commutative field \mathbb{K} and a non-commutative convolution product $*$; subsequently, it has been built up a (trivial) structure of braided non-commutative Hopf algebra on it.

We claim that this last (albeit trivial) Hopf structure is the first and most natural possible one, on such a groupoid algebra, because of the no (algebraic) finiteness of this generated algebra (since $\text{card } I = \infty$, in general).

Therefore, the main interest of the paper [Iu], must be searched in the physical construction of the EBJ EBB-groupoid.

In this paper, we'll try to build up other (less trivial) structures on this special HBJ EBB-groupoid, through adapted methods and tools of the theory of quantum groups, relative both to the infinite-dimensional case and finite-dimensional case¹.

Furthermore, these structures will be introduced taking into account eventual physical motivations.

The (above mentioned) natural structure of Hopf algebra on $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$ is given as follows: coproduct $\Delta(x) = x \otimes x$, counit $\varepsilon(x) = 1$, and antipode the extended inversion map.

As already said, the first, natural structure of a braided (or quasitriangular) non-commutative Hopf algebra on such an algebra, is trivially given by the universal R-matrix $R = 1 \otimes 1$, whence a (trivial) example of quantum group if one assume a braided (or quasitriangular) non-commutative Hopf algebra as definition of quantum group. Instead, a non-trivial example of quasitriangular Hopf algebra arises from Drinfeld quantum double constructions (see § 6.).

There exists other definitions of a quantum group structure: for instance, if we consider a non-commutative and non-cocommutative Hopf algebra as quantum group, then a cross (or bicross, or double cross) product construction may provide examples of such a quantum group, whereas, if we consider as special 'quantum objects' the result of a non-degenerate dual pairing of Hopf algebras, then a Heisenberg double may be taken as an example of quantum group.

If one want to determine examples of these last structures starting from $\mathcal{G}_{HBJ}(\mathcal{F}_I)$, it is necessary, at first, examines the possible dual structures of $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$, taking into account the existence of some problems for this

¹The most interesting case, from the physical view-point, is that finite-dimensional corresponding to $\text{card } I < \infty$, since any physical spectroscopic system has a finite number of energy levels.

particular case study.

The first problem (that we'll sketch at the paragraph 3.) is related to dualization in the infinite-dimensional case, whereas the second problem², because of the infinity of $\mathcal{G}_{HBJ}(\mathcal{F}_I)$, is due to the tentative of giving a Hopf algebra structure to the \mathbb{K} -algebra of \mathbb{K} -valued functions defined on the HBJ EBB-groupoid $\mathcal{G}_{HBJ}(\mathcal{F}_I)$, say $\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$: in fact, on this algebra (that is strictly correlated to the first problem of dualization of $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$, and viceversa) it is a problematic question to define the right comultiplication and counit, for the following reasons.

For a group (\mathcal{G}, \cdot) , the comultiplication question do not subsist in the finite-dimensional case, because of the natural identification

$$\mathcal{F}_{\mathbb{K}}(\mathcal{G}) \otimes \mathcal{F}_{\mathbb{K}}(\mathcal{G}) \cong \mathcal{F}_{\mathbb{K}}(\mathcal{G} \times \mathcal{G});$$

in such a case, a natural structure of Hopf algebra on $\mathcal{F}_{\mathbb{K}}(\mathcal{G})$, is given by the following data :

1. coproduct $\Delta : \mathcal{F}_{\mathbb{K}}(\mathcal{G}) \rightarrow \mathcal{F}_{\mathbb{K}}(\mathcal{G} \times \mathcal{G})$, given by $\Delta(f)(g_1, g_2) = f(g_1 \cdot g_2)$, for all $g_1, g_2 \in \mathcal{G}$;
2. counit $\varepsilon : \mathcal{F}_{\mathbb{K}}(\mathcal{G}) \rightarrow \mathbb{K}$, with $\varepsilon(f) = 1$;
3. antipode $S : \mathcal{F}_{\mathbb{K}}(\mathcal{G}) \rightarrow \mathcal{F}_{\mathbb{K}}(\mathcal{G})$, defined as $S(f)(g) = f(g^{-1})$ for all $g \in \mathcal{G}$,

where the functional laws in 1. and 2. are well-defined since, respectively, the group law is totally defined in \mathcal{G} , and there exists a unique unit.

Instead, if we consider a generic groupoid, these two questions remains unsolved, in the finite-dimensional case too, both for the partial definition of the groupoid law and for the existence of many unities: for these reasons, the initial definitions 1. and 2. of above, are ill-posed in this case.

Nevertheless, it is possible to solve these last problems with some extensions in the above definitions, remaining in the context of classical Hopf algebra theory, but with minor usefulness of results.

Instead, in the new realm of the extended Hopf algebra structures, this problem may be clarified and solved with fruitfulness, at least for the dual $\mathcal{F}_{\mathbb{K}}^*(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \subseteq \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$.

²In a certain sense, preliminary to the first one.

2. Cross product algebras

The notions of cross product and bicrossproduct are important tools for an algebraic setting of some fundamental structures of Quantum Mechanics. In this paper, we'll do only with the notion of cross (or smash, or semidirect) product.

Let $V_{\mathbb{K}}$ be a \mathbb{K} -linear space, A a \mathbb{K} -algebra and $\psi : A \otimes V \rightarrow V$ a \mathbb{K} -linear map; if we pose $\psi(h \otimes v) = \psi_h(v)$, and if $\psi_{ab}(v) = \psi_a(\psi_b(v))$, $\psi_1(v) = v$, $\forall a, b \in A, \forall v \in V$, then $(A, V_{\mathbb{K}}, \psi)$ is a *left A -module* on $V_{\mathbb{K}}$; we say that $(A, V_{\mathbb{K}}, \psi)$ is a *left action* of A on $V_{\mathbb{K}}$, or that $V_{\mathbb{K}}$ is a *left A -module*.

Usually, we write $a \triangleright v$ instead of $\psi_a(v)$, so that the action axioms are write as $(ab) \triangleright v = a \triangleright (b \triangleright v)$ and $1 \triangleright v = v$.

If A is a Hopf algebra, $V_{\mathbb{K}}$ is an A -module algebra [coalgebra], and $a \triangleright (vw) = (a_{(1)} \triangleright v)(a_{(2)} \triangleright w)$, $a \triangleright 1_V = \varepsilon(h)1_V$ [$\Delta(a \triangleright v) = (a_{(1)} \triangleright v_{(1)}) \otimes (a_{(2)} \triangleright v_{(2)})$] (that is to say $\Delta(a \triangleright v) = \Delta_A(a) \triangleright \Delta(v)$), $\varepsilon(a \triangleright v) = \varepsilon(a)\varepsilon(v)$ for all $v, w \in V_{\mathbb{K}}$ and $a \in A$, then $V_{\mathbb{K}}$ is said a *left A -module algebra* [coalgebra].

If V is a Hopf algebra [bialgebra], then there exists the following two natural left actions on itself: the *left regular action* L , given by $L_v(w) = vw$, and the *left adjoint action* Ad , given by $Ad_v(w) = v_{(1)}wS(v_{(2)})$, for all $v, w \in V$.

The *left coregular action* R^* of a finite-dimensional Hopf algebra [bialgebra] V on the dual V^* , is given by $R_v^*(\phi) = \phi_{(1)}\langle v, \phi_{(2)} \rangle$, whereas, in the infinite-dimensional case, we set $\langle R_v^*(\phi), w \rangle = \langle \phi, vw \rangle$, for all $v, w \in V$ and $\phi \in V^*$, being \langle , \rangle the dual pairing between V and V^* (V^o in the infinite-dimensional case); furthermore, R^* makes V^* [V^o] into a V -module algebra.

The *left coadjoint action* of a finite-dimensional Hopf algebra [bialgebra] V on the dual V^* , is given by $Ad_v^*(\phi) = \phi_{(2)}\langle v, (S\phi_{(1)})\phi_{(2)} \rangle$, whereas, in the infinite-dimensional case, we put $\langle Ad_v^*(\phi), w \rangle = \langle \phi, (Sv_{(1)})wv_{(2)} \rangle$, for all $v, w \in V$ and $\phi \in V^*$, being \langle , \rangle the dual pairing between V and V^* (V^o in the infinite-dimensional case); furthermore, Ad^* makes V^* [V^o] into a V -module coalgebra.

The concept of A -module algebra generalizes³ the notion of G -covariant algebra of the Physics: if G is a symmetry group, given a G -covariant \mathbb{K} -algebra V , we construct the group algebra $\mathbb{K}G$ generated by G ; then, the algebra generated by $\mathbb{K}G$ and V , with commutation relations given by $uv = (u \triangleright v)u \quad \forall v \in V, u \in G$, give rise to a semidirect, or cross, product algebra.

³Because the structure of A -module (from commutative algebra) generalize the notion of representation.

Therefore, we have the following general structure.

Given a Hopf algebra [bialgebra] A and a left A -module algebra on V , then there exists a *left cross product* algebra on $V \otimes A$, with product

$$(v \otimes a)(w \otimes b) = v(a_{(1)} \triangleright w) \otimes a_{(2)}b, \quad v, w \in V, \quad a, b \in A$$

and unit element $1 \otimes 1$. This algebra is denoted with⁴ $V \rtimes A$.

With obvious modifications, it is possible to have right actions, as follows.

A *right action* of an algebra A on the \mathbb{K} -linear space $V_{\mathbb{K}}$, is a linear map $V \otimes A \rightarrow V$, denoted by $v \otimes a \rightarrow v \triangleleft a$, such that $v \triangleleft (ab) = (v \triangleleft a) \triangleleft b$ and $v \triangleleft 1 = v$ for all $a, b \in A$, $v \in V$. When A is a Hopf algebra that acts at right on an algebra V , and $(ab) \triangleleft v = (a \triangleleft v_{(1)})(b \triangleleft v_{(2)})$, $1_V \triangleleft v = 1_V \varepsilon(v)$, for all $a, b \in A$ and $v \in V$, then we say that V is a *right A -module algebra*, and we write (V, A, \triangleleft) .

A *right A -coaction* on a \mathbb{K} -linear space $V_{\mathbb{K}}$, is a linear map $\varphi : V \rightarrow V \otimes A$ such that $(\varphi \otimes \text{id}) \circ \varphi = (\text{id} \otimes \Delta) \circ \varphi$ and $\text{id} = (\text{id} \otimes \varepsilon) \circ \varphi$; we say, also, that $V_{\mathbb{K}}$ is a *right A -comodule*.

If we set $\varphi(v) = v^{(1)} \otimes v^{(2)}$ (in $V \otimes A$), then a coalgebra $V_{\mathbb{K}}$ is a *right A -comodule coalgebra* if $V_{\mathbb{K}}$ is a right A -comodule and

$$v_{(1)}^{(1)} \otimes v_{(2)}^{(1)} \otimes v^{(2)} = v_{(1)}^{(1)} \otimes v_{(2)}^{(1)} \otimes v_{(1)}^{(2)} v_{(2)}^{(2)}, \quad \varepsilon(v^{(1)})v^{(2)} = \varepsilon(v).$$

If V is a Hopf algebra [bialgebra], there are two natural right actions on itself: the *right regular action* R , given by $R_v(w) = wv$, and the *right adjoint action* Ad , given by $Ad_v(w) = (Sv_{(1)})wv_{(2)}$, for all $v, w \in V$.

If A is a Hopf algebra [bialgebra] and V is a right A -comodule coalgebra, then there exists a *right cross coproduct* coalgebra structure on $A \otimes V$, given by

$$\Delta(a \otimes v) = a_{(1)} \otimes v_{(1)}^{(1)} \otimes a_{(2)} v_{(1)}^{(2)} \otimes v_{(2)}, \quad \varepsilon(a \otimes v) = \varepsilon_A(a) \varepsilon(v),$$

for all $a \in A$, $v \in V$; such a coalgebra, is denoted with⁵ $A \ltimes V$.

We do not discuss the notion of bicrossproduct algebra, introduced by S. Majid in connections with an important tentative of unifying Quantum mechanics and Gravity (see [Ma], Chap. 6), but we give only a simple example of what a bicrossproduct Hopf algebra is: let G, M two subgroups that factorizes a given group, so that G acts on M , and viceversa (for instance, M may be the position space, while G may be the momentum group); let $\mathbb{K}(M)$

⁴Or with $V \rtimes_{\psi} A$, if one want to specify the underling left action ψ .

⁵Or with $V \ltimes_{\psi} A$, if one want to specify the underling right action ψ .

the algebra of \mathbb{K} -valued functions on M , and let $\mathbb{K}G$ be the free algebra on G . Then, the following Hopf algebra

$$\mathbb{K}(M) \bowtie \mathbb{K}G = \begin{cases} \mathbb{K}(M) \rtimes \mathbb{K}G & \text{as algebra,} \\ \mathbb{K}(M) \rtimes \mathbb{K}G & \text{as coalgebra} \end{cases}$$

is a first example of bicrossproduct algebra, whose dual Hopf algebra is $\mathbb{K}M \bowtie \mathbb{K}(G)$.

Another notion strictly correlated to that of bicrossproduct, is the notion of *double cross product* (see [Ma]).

The cross, bicross and double cross product constructions, provides a large class of quantum groups.

3. The restricted Hopf algebra structure on $\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$

There exists various methods to define a classical Hopf algebra structure on $\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$, recalling that this algebra is infinite-dimensional.

The main method, proceed as follows.

In the case of the groupoid $\mathcal{G}_{HBJ}(\mathcal{F}_I)$, that we recall to be a particular example of the general type $(G, G^{(0)}, r, s, \star)$, the most natural modifications to the functional laws on the points 1. and 2. of the § 1, are the following (see [Va], § 2.2):

- 1'. coproduct: $\Delta(f)(g_1, g_2) = f(g_1 \star g_2)$ if $(g_1, g_2) \in G^{(2)}$, and $= 0$ otherwise;
- 2'. counit: $\varepsilon(f) = \sum_{e \in G^{(0)}} f(e)$.

The antipode definition 3., is the same also in this case.

Besides the question relative to the functional laws, there exists the question related to their definition sets.

Since, in the infinite-dimensional case we have

$$\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \otimes \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \subseteq \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I) \times \mathcal{G}_{HBJ}(\mathcal{F}_I))$$

with

$$\Delta : \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \rightarrow \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I) \times \mathcal{G}_{HBJ}(\mathcal{F}_I)),$$

let $\mathcal{F}^o = \Delta^{-1}(\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \otimes \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))) \subseteq \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$; then, \mathcal{F}^o is a Hopf algebra with the coalgebra structure given by 1', 2'. and 3., although it is difficult to determine exactly its set-theoretic specificity.

It is called the *restricted Hopf algebra* of $\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$, and is denoted with $\mathcal{F}_{\mathbb{K}}^o(\mathcal{G}_{HBJ}(\mathcal{F}_I))$.

For our purpose, in the finite-dimensional case, there exists the isomorphism (see [Ks], III.1; [Ma], Example 1.5.4) $\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \cong \mathcal{A}_{\mathbb{K}}^*(\mathcal{G}_{HBJ}(\mathcal{F}_I))$, so that we may construct a Hopf algebra structure on $\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$ via $\mathcal{A}_{\mathbb{K}}^*(\mathcal{G}_{HBJ}(\mathcal{F}_I))$ by dual pairing; unfortunately, this isomorphism do not subsists in the infinite-dimensional case, and such a question will be at the basis of the discussion of § 6.

Other methods for dualization (as Konstant duality, Cartier duality, Tannaka-Krein duality, Takeuchi duality, Kadison-Szlachányi dual pairing, the weak antipode plus convolution-inverse method, Pontryagin duality, and so on), may be found, for instance, in [Sw], [Sch1], [Sch2], [Sch3], [Ma].

However, in the context of the classical Hopf algebra structures, some of these methods do not lead to an explicit solution of the problem, while others provides complicated structures unadapted to the physical applications. But there exists different generalizations of the structure of Hopf algebra (for a recent survey of these, see [Ka]) as, for instance, the notions of weak Hopf algebra (or quantum groupoid) and Hopf algebroid (see [BNS], [NV]), through which it is possible to solve, more explicitly, the above problem, at least for the dual $\mathcal{A}_{\mathbb{K}}^*(\mathcal{G}_{HBJ}(\mathcal{F}_I))$, in the context of weak Hopf algebras, and with more possibilities on the side of physical applications. Such a question, we'll be the matter of a further paper.

4. The restricted Heisenberg double $\mathcal{H}_{\mathcal{A}}^o(\mathcal{G}_{HBJ}(\mathcal{F}_I))$

Hence, as regard what has been said above, we may consider the following dual pairing

$$\langle \cdot, \cdot \rangle : \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \times \mathcal{F}_{\mathbb{K}}^o(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \rightarrow \mathbb{K}$$

such that

$$\begin{aligned} \langle a_1 \otimes a_2, \Delta_{\mathcal{F}^o}(f) \rangle &= \langle a_1 a_2, f \rangle, & \langle \Delta_{\mathcal{A}}(a), f_1 \otimes f_2 \rangle &= \langle a, f_1 f_2 \rangle \\ \langle 1_{\mathcal{A}}, f \rangle &= \varepsilon_{\mathcal{F}^o}(f), & \langle 1_{\mathcal{F}^o}, a \rangle &= \varepsilon_{\mathcal{A}}(a) \end{aligned}$$

for all $f, f_1, f_2 \in \mathcal{F}^o$, and $a, a_1, a_2 \in \mathcal{A}$.

It is know that it is always possible to consider, eventually quotienting, a non-degenerate dual pairing of this type. Therefore, if we consider the action

$$(b, a) \rightarrow b \triangleright a = \langle b, a_{(1)} \rangle a_{(2)} \quad \forall a \in \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)), \quad \forall b \in \mathcal{F}_{\mathbb{K}}^o(\mathcal{G}_{HBJ}(\mathcal{F}_I)),$$

it follows that it is possible to define the left cross product algebra

$$\mathcal{H}_{\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)), \mathcal{F}_{\mathbb{K}}^o(\mathcal{G}_{HBJ}(\mathcal{F}_I))} = \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \rtimes \mathcal{F}_{\mathbb{K}}^o(\mathcal{G}_{HBJ}(\mathcal{F}_I)),$$

called the *Heisenberg double* of the pair $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)), \mathcal{F}_{\mathbb{K}}^o(\mathcal{G}_{HBJ}(\mathcal{F}_I))$. This last construction may be repeated for the restricted dual $\mathcal{A}_{\mathbb{K}}^o(\mathcal{G}_{HBJ}(\mathcal{F}_I)) (\subseteq \mathcal{A}_{\mathbb{K}}^*(\mathcal{G}_{HBJ}(\mathcal{F}_I)))$ of $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$, obtaining the so-called *Heisenberg double* of $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$, that we denote, for simplicity, with $\mathcal{H}_{\mathcal{A}}^o(\mathcal{G}_{HBJ}(\mathcal{F}_I))$.

5. The restricted Weyl algebra

The notion of cross product lead to an algebraic formulation of some aspects of quantization.

Let V be a A -module algebra, with A a Hopf algebra, and let $V \rtimes A$ be the corresponding left cross product. Hence, there exists a canonical representation on V itself, given by $(v \otimes a) \triangleright w = v(a \triangleright w)$, called the *generalized Schrödinger representation* of V .

The physical motivations to this terminology arise from the quantum meaning that such a representation has when applied to the bicrossproduct algebra $\mathbb{K}(M) \rtimes \mathbb{K}G$ of the end of paragraph 2 (see also [Ma], Chap. 6).

Our interest is on infinite-dimensional case⁶, so let V be a infinite-dimensional Hopf algebra, with restricted dual V^o ; then, by the left coregular representation R^* of V on V^o (that does holds also in the infinite-dimensional case, as seen at § 2.), V^o is a V -module algebra, so that we may consider the left cross product algebra $V^o \rtimes V$.

Nevertheless, we are interested to another type of left cross product algebra, built up as follows (see [Ma], § 6.1, for details).

We consider the following action $\phi \triangleright v = v_{(1)} \langle \phi, v_{(2)} \rangle$ for all $v \in V, \phi \in V^o$, making V into a V^o -module algebra and that gives rise to the following product on $V \otimes V^o$

$$(v \otimes \phi)(w \otimes \psi) = vw_{(1)} \otimes \langle w_{(2)}, \phi_{(1)} \rangle \phi_{(2)} \psi,$$

whence a structure of left cross product algebra on $V \otimes V^o$, namely $V \rtimes V^o$. Then, it is possible to prove that the related Schrödinger representation give rise to an isomorphism (of algebras) $\chi : V \rtimes V^o \rightarrow \text{Lin}(V)$, where $\text{Lin}(V)$ is

⁶But not only; for example, in the finite-dimensional case, we have $V^o = V^*$, and what follows holds also in this case, with obvious modifications.

the algebra of \mathbb{K} -endomorphisms of V , given by $\chi(v \otimes \psi)w = vw_{(1)}\langle\phi, w_{(2)}\rangle$. Therefore, we have the algebra isomorphism $\mathcal{W}(V) = V \rtimes V^o \cong \text{Lin}(V)$; we call $\mathcal{W}(V)$ the *restricted Weyl algebra* of the Hopf algebra V .

This last construction is an algebraic generalization of the usual Weyl algebras of Quantum Mechanics on a group, whose finite-dimensional prototype is as follows.

We consider the strict dual pair given by the \mathbb{K} -valued functions on G , say $\mathbb{K}(G)$, and the free algebra on G , say $\mathbb{K}G$; then, the left cross product algebra $\mathbb{K}(G) \rtimes \mathbb{K}G$ is given by the right action of G on itself, namely $\psi_u(s) = su$, that induces a left regular representation of G on $\mathbb{K}G$, hence a Schrödinger representation generated by this and by the action of $\mathbb{K}G$ on itself by point-wise product. Whence, if $V = \mathbb{K}(G)$, we obtain the left cross product algebra $\mathbb{K}(G) \rtimes \mathbb{K}G$, isomorphic to $\text{Lin}(\mathbb{K}(G))$ by Schrödinger representation, that formalizes the algebraic quantization of a particle moving on G by translations.

We may apply these well-known considerations (see [Ma]) to $\mathcal{G}_{HBJ}(\mathcal{F}_I)$ when $\text{card } I < \infty$ (finite number of energy levels), taking into account that (in the finite-dimensional case) $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) = \mathcal{F}_{\mathbb{K}}^*(\mathcal{G}_{HBJ}(\mathcal{F}_I))$, in such a way that the Weyl algebra

$$\begin{aligned} \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \rtimes \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) &= \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \rtimes \mathcal{F}_{\mathbb{K}}^*(\mathcal{G}_{HBJ}(\mathcal{F}_I)) = \\ &= \mathcal{W}(\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))) \cong \text{Lin}(\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))) \end{aligned}$$

represents the algebraic quantization of a particle moving on the groupoid $\mathcal{G}_{HBJ}(\mathcal{F}_I)$ by translations⁷.

Instead, for the infinite-dimensional HBJ EBB-groupoid $\mathcal{G}_{HBJ}(\mathcal{F}_I)$, we obtain a particular restricted Weyl algebra of the following type

$$\mathcal{W}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) = \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \rtimes \mathcal{F}_{\mathbb{K}}^o(\mathcal{G}_{HBJ}(\mathcal{F}_I))$$

with $\mathcal{F}_{\mathbb{K}}^o(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \neq \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$ because of the no finitely generation of $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$; therefore, the above physical interpretation of the restricted Weyl algebra $\mathbb{K}(G) \rtimes \mathbb{K}G$, is no longer valid for $\mathcal{W}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$.

However, as we'll see in another place, the last lack of physical interpretation can be restored in the context of extended Hopf algebra structures.

6. The Drinfeld quantum double

⁷This remark may be think as the starting point for a quantum mechanics on a groupoid.

If V is a Hopf algebra, then through the left adjoint action (in infinite-dimensional setting) Ad on itself, we have that V is a V -module algebra, so that we may build the left cross product algebra $V \rtimes_{Ad} V$.

We consider the right adjoint action of $\mathcal{G}_{HBJ}(\mathcal{F}_I)$ on itself, given by $\psi_g(h) = g^{-1} \star h \star g$ if exists, $= 0$ otherwise; such an action makes $\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$ into a $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$ -module algebra.

In the finite-dimensional case we have $\mathcal{A}_{\mathbb{K}}^*(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \cong \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$, hence

$$\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) = \mathcal{A}_{\mathbb{K}}^{**}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \cong \mathcal{F}_{\mathbb{K}}^*(\mathcal{G}_{HBJ}(\mathcal{F}_I)),$$

so that $\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$ is also a $\mathcal{F}_{\mathbb{K}}^*(\mathcal{G}_{HBJ}(\mathcal{F}_I))$ -module algebra, whence the left cross product algebra

$$\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \rtimes \mathcal{F}_{\mathbb{K}}^*(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \cong \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \rtimes \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$$

that the tensor product coalgebra makes into a Hopf algebra, called the *quantum double* of $\mathcal{G}_{HBJ}(\mathcal{F}_I)$ and denoted with $D(\mathcal{G}_{HBJ}(\mathcal{F}_I))$; even in the finite-dimensional case, it represent the algebraic quantization of a particle constrained to move on conjugacy classes of $\mathcal{G}_{HBJ}(\mathcal{F}_I)$ (quantization on homogeneous spaces over a groupoid).

Besides, it was proved, for a finite group G , that this (Drinfeld) quantum double $D(G)$ has a quasitriangular structure (see [Ma], Chap. 6), given by

$$(\delta_s \otimes u)(\delta_t \otimes v) = \delta_{u^{-1}s u, t} \delta_t \otimes uv, \quad \Delta(\delta_s \otimes u) = \sum_{ab=s} \delta_a \otimes u \delta_b \otimes u,$$

$$\varepsilon(\delta_s \otimes u) = \delta_{s,e}, \quad S(\delta_s \otimes u) = \delta_{u^{-1}s^{-1}u} \otimes u^{-1},$$

$$R = \sum_{u \in G} \delta_u \otimes e \otimes 1 \otimes u,$$

where we have identifies the dual of $\mathbb{K}G$ with $\mathbb{K}(G)$ via the idempotents p_g , $g \in G$ such that $p_g p_h = \delta_{g,h} p_g$ (see [NV], 2.5. and [Ma], 1.5.4); such a quantum double represents the algebra of quantum observables of a certain physical system.

Hence, it is a natural question to ask if such structures may be extended to $D(\mathcal{G}_{HBJ}(\mathcal{F}_I))$, that is when we have a quantum double built on a groupoid.

7. Conclusions.

From what has been said above (and in [Iu]), it can be meaningful to study as the common known structures, just described above, may be extended when we consider a groupoid⁸ instead of a finite group, both in the classical theory

⁸Finite or not.

of Hopf algebras and in the new realm of the extended Hopf structures. Subsequently, the resulting structures must be interpreted from the physical view point, with a critical comparison respect to the physical meaning of the classical Hopf structures just seen in this paper.

These questions are not trivial, because there are recent papers on a classical Physics on a groupoid (see, for instance, [CDMMM]), and many important works on the role of Hopf algebras in High Energy Physics⁹ (see, for instance, [Kr] and references therein).

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⁹Perhaps, it should be interesting to apply the extended Hopf algebra structures to this context.

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